American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2009-2010

Final Exam - solution

Exercise 1 **a.** Find the directional derivative of $f(x, y) = x^2e^{-2y}$ at $P(1, 0)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

b. The equation $1 - x - y^2 - \sin(xy)$ defines y as a differentiable function of x. Find dy/dx at the point $P(0, 1)$.

c. Find the points on the surface $xy + yz + zx - x - z^2 = 0$, where the tangent plane is parallel to the xy-plane.

Exercise 2 Find the absolute minimum and maximum values of $f(x, y) = x^2+xy+y^2-3x+3y$ on the triangular region R cut from the first quadrant by the line $x + y = 4$.

Exercise 3 Use Lagrange Multipliers to find the maximum and the minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 10$.

Exercise 4 Reverse the order of integration, then evaluate the integral

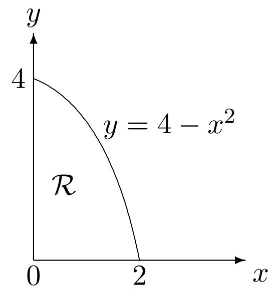
$$I = \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx$$

<u>*solution*</u>: to express the integral in the order $dxdy$, we sketch the region of integration $\mathcal{R}$ in the xy-plane.

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

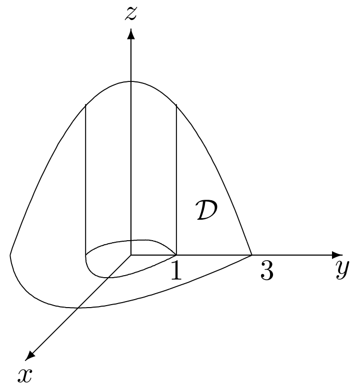
$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^4 \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{8}\right]_0^4 = \frac{e^8-1}{4}$$



Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the paraboloid $z = 9 - x^2 - y^2$, and lying outside the cylinder $x^2 + y^2 = 1$.

<u>*solution*</u>:

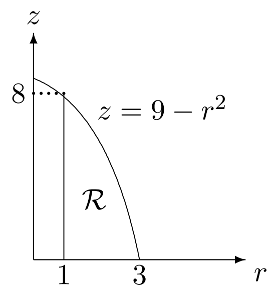


$$a)\ V = \int_0^{2\pi}\int_1^3\int_0^{9-r^2} r\,dz dr d\theta$$

$$= 2\pi \int_1^3 r(9 - r^2)\,dr$$

$$= 2\pi \left[\tfrac{9}{2} r^2 - \tfrac{r^4}{4}\right]_1^3 = 32\pi$$

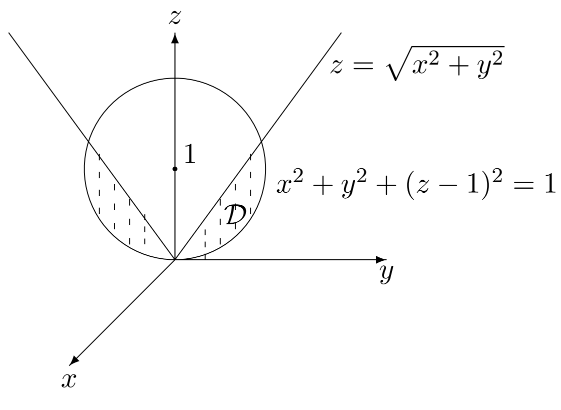
b) to express the integral in the order $drdzd\theta$, we sketch the region of integration $\mathcal{R}$ in the rz-plane.



$$V = \int_0^{2\pi}\int_0^{8}\int_1^{\sqrt{9-z}} r\,dr\,dz\,d\theta$$

Exercise 6 Let V be the volume of the region that is bounded form below by the sphere $x^2 + y^2 + (z-1)^2 = 1$ and from above by the cone $z = \sqrt{x^2+y^2}$. Express V as an iterated triple integral in spherical coordinates, then evaluate the resulting integral *(sketch the region of integration)*.

<u>*solution*</u>: The equation of the sphere $x^2 + y^2 + (z-1)^2 = 1$ in spherical coordinates is $\rho = 2\cos\phi$



$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{8}{3}\cos^3\phi \sin\phi\ d\phi\ d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}\Big[-\cos^4\phi\Big]_{\pi/4}^{\pi/2} d\theta = \int_0^{2\pi}\frac{1}{6}\ d\theta = \frac{\pi}{3}$$

Exercise 7 Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$.
(do not evaluate any of the integrals)

Exercise 8 a. Find the line integral of $f(x,y) = (x+y^2)/\sqrt{1+x^2}$ along the curve $C : y = x^2/2$ from $(1, 1/2)$ to $(0,0)$.

solution: - $r(t) = (1-t)\mathbf{i} + \frac{(1-t)^2}{2}\mathbf{j}$, $0 \le t \le 1$;

- $v(t) = \frac{dr}{dt} = -\mathbf{i} - (1-t)\mathbf{j}$, and $|v(t)| = \sqrt{1+(1-t)^2}$;

- $f(t) = \frac{(1-t)+\frac{(1-t)^4}{4}}{\sqrt{1+(1-t)^2}}$, hence

$$\int_C f(s)ds = \int_0^1 f(t).|v(t)|dt = \int_0^1 \left[(1-t) + \frac{(1-t)^4}{4}\right] dt = 11/20$$

b. Show that the differential form $2\cos y dx + (\frac{1}{y} - 2x\sin y)dy + (1/z)dz$ is exact, then evaluate

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz$$

solution: $f(x,y,z) = 2x\cos y + \ln(yz) + C$ is a potential function (check it!), hence

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz = [2x\cos y + \ln(yz) + C]_{(0,2,1)}^{(1,\pi/2,2)} = \ln(\pi/2)$$

c. Find the *counterclockwise circulation* of the field $F = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $x = 1$ and the curve $y = x^3$.

i) direct calculation: *cicculation* $= \oint_C Mdx + Ndy$

$C_1 : r_1(t) = t\mathbf{i}$, $0 \le t \le 1$; $Mdx + Ndy = 0$, and $\int_{C_1} dx + Ndy = 0$

$C_2 : r_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; $Mdx + Ndy = 4t^2dt$, and $\int_{C_2} Mdx + Ndy = \int_0^1 4t^2 dt = 4/3$

$C_3 : r_3(t) = (1-t)\mathbf{i} + (1-t)^3\mathbf{j}$, $0 \le t \le 1$; $Mdx + Ndy = -14(1-t)^{10}dt$, and

$$\int_{C_3} Mdx + Ndy = \int_0^1 -14(1-t)^{10}dt = -14/11$$

$$circulation(F) = \oint_C Mdx + Ndy = 0 + 4/3 - 14/11 = 2/33$$

ii) Green's theorem: $circulation(F) = \int\int_R (\mathbf{curlF}) \cdot \mathbf{k}\, dA$

$$(\mathbf{curlF}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 8xy^2 - 6xy^2 = 2xy^2$$

$$\int\int_R (\mathbf{curlF}) \cdot \mathbf{k}\, dA = \int_0^1 \int_0^{x^3} 2xy^2\, dy\, dx = \int_0^1 2x \left[\frac{y^3}{3}\right]_0^{x^3} dx = 2/3 \int_0^1 x^{10} dx = 2/33$$